LONGITUDINAL FLOW PAST A CYLINDER IMMERSED IN AN INHOMOGENEOUS FLOW IN THE PRESENCE OF STEADY BOUNDARY-LAYER SUCTION

(PRODOL'NOE OVTEKANE CILINDRA NEODNORODNYM POTOKOM PRI NALICHII USTANOVIVSHEGOCIA POGRANICHNOVO SLOIA)

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O. N. OVCHINNIKOV
(Leningrad)

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An exact solution is presented of the problem of the velocity and temperature distributions near a cylinder placed lengthwise in the stream of a viscous incompressible fluid, the velocity of which is given in the form $U = U_0 + \omega_0 r^2$, in the presence of distributed boundary-layer suction.

1. The dynamic boundary layer on the cylinder. We shall investigate the problem of the distribution of velocities and temperatures near a heated cylinder of radius a placed lengthwise in an inhomogeneous stream of a viscous incompressible fluid, the velocity of which is given in the form

$$U = U_0 + \omega_0 r^2 \qquad (r \geqslant a) \tag{1.1}$$

where U_0 and ω_0 are constants, in the presence of steady boundary-layer suction, the intensity of which is assumed constant.

If we assume that the flow is steady and axially symmetrical and that the velocity components and temperature distributions are independent of the z coordinate directed along the cylinder axis in the direction of the flow, then the Navier-Stokes equations have the form

$$v_r \frac{dv_r}{dr} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rv_r)\right], \qquad \frac{d(rv_r)}{dr} = 0$$
$$v_r \frac{dv_z}{dr} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\nu}{r}\frac{d}{dr}\left[r\frac{dv_z}{dr}\right] \tag{1.2}$$

$$\rho g c_p v_r \frac{dT}{dr} = \frac{\lambda}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + 2\mu\left[\left(\frac{dv_r}{dr}\right)^2 + \left(\frac{v_r}{r}\right)^2 + \frac{1}{2}\left(\frac{dv_z}{dr}\right)^2\right] \tag{1.3}$$

where λ is the coefficient of heat conductivity, g is the gravitational constant, ν is the kinematic viscosity, ρ is the density, μ is the absolute viscosity, c_p is the specific heat of the medium, p is the pressure, and u and v are the velocity components along the z- and r-axes, respectively. It is easy to see that the solution of the first three equations of this system that satisfy the boundary conditions

$$v_z = 0, \quad v_r = v_0 < 0 \quad \text{for } r = a, \qquad v_z \to U_0 + \omega_0 r^2, \quad v_r \to 0 \quad \text{for } r \to \infty \tag{1.4}$$

has the form

$$\frac{v_z}{U_0} = 1 - R^k + \beta (R^2 - R^k), \qquad \frac{p - p_0}{\rho U_0^2} = 2(2-k)\frac{\beta z_1}{\mathrm{Re}} - \frac{m^2}{2R^2}, \qquad \frac{v_r}{U_0} = \frac{m}{R} \tag{1.5}$$

where for abbreviation the following notation is introduced:

$$R = \frac{r}{a}, \quad z_1 = \frac{z}{a}, \quad \beta = \frac{\omega_0 a^2}{U_0}, \quad m = \frac{v_0}{U_0}, \quad \mathrm{Re} = \frac{U_0 a}{\nu}, \quad k = m\,\mathrm{Re} \tag{1.6}$$

and p_0 is an arbitrary constant.

The shear stress due to friction at the wall r_0 is determined by the following relation:

$$\tau_0 = \mu \frac{dv_z}{dr}\bigg|_{r=a} = \mu \frac{U_0}{a}[\beta(2-k) - k] \tag{1.7}$$

Hence, it follows immediately that the flow remains attached to the circular cylinder only if the condition $\beta(2 - k) - k > 0$ is fulfilled, i.e. for a particular relationship between the exhaust velocity and the order of inhomogeneity of the oncoming stream.

Usually, boundary-layer suction is used to diminish the friction resistance by moving the transition point from laminar to turbulent boundary layer downstream. Consequently, in practice, large Reynolds numbers are of main interest, i.e. large k values (absolute magnitude). Then it follows from (1.5) and (1.7) that even a comparatively weak inhomogeneity of the oncoming stream will have a noticeable influence on the distribution of velocities and the shear stress at the wall. This is illustrated in the figure, in the upper part of which are shown the profiles of the velocities v_z/U_0 in the boundary layer at the cylinder, calculated from Formula (1.5) for $k = 100$, for the three values of vorticity of the oncoming stream $\beta = -0.1$, 0 and 0.1.

2. Thermal boundary layer on the cylinder. If we substitute (1.5) and (1.3) and take into account (1.6), then we obtain the following differential equation for the determination of temperature distribution near a heated cylinder:

$$\frac{1}{R}\frac{d}{dR}\left(R\frac{d\vartheta}{dR}\right) - \frac{k\sigma}{R}\frac{d\vartheta}{dR} = -2\sigma\left[\frac{4m^2}{R^4} + 4\beta^2 R^2 - 4\beta(1+\beta)kR^k + (1+\beta)^2 k^2 R^{2k-2}\right] \tag{2.1}$$

the solution of which depends on definite heat boundary conditions; here

$$\vartheta = \frac{T}{U_0^2 / 2gc_p}, \qquad \sigma = \frac{\mu g c_p}{\lambda} \quad \text{(Prandtl Number)} \tag{2.2}$$

The thermometer problem. If we assume that the conduction of heat from the wall to the fluid is zero, i.e. $d\vartheta/dR = 0$ for $R = 1$, then the solution of Equation (2.1) which satisfies this condition may be written in the following form:

$$\vartheta - \vartheta_0 = \frac{2\sigma\beta^2}{4-k\sigma}\left[\frac{4}{\sigma k}R^{\sigma k} - R^4\right] - \frac{4m^2\sigma}{2+k\sigma}\left[\frac{2}{\sigma k}R^{\sigma k} - R^{-2}\right] + $$
$$+ \frac{8\beta(1+\beta)}{k+2-k\sigma}\left[\frac{k\sigma}{k+2}R^{k+2} - R^{k\sigma}\right] + \frac{\sigma(1+\beta)^2}{2-\sigma}\left[\frac{2}{\sigma}R^{k\sigma} - R^{2k}\right] \tag{2.3}$$

where ϑ_0 is some characteristic temperature.

From the last equation it is easy to see that at a sufficient distance from the cylinder there exists a cross-stream temperature drop of the following magnitude:

$$\vartheta_\infty = \vartheta_0 - \frac{4\sigma\beta^2}{4-k\sigma}R^4 - \frac{4\sigma m^2}{2+k\sigma}R^{-2} \tag{2.4}$$

i.e. an inhomogeneity of velocity field in an oncoming stream causes the inhomogeneity of the temperature field. We shall now determine the proper temperature of the wall T_e. Assuming $R = 1$ in (2.3) and taking into account (1.6) and (2.2), we have

$$T_e - T_0 = \frac{U_0^2}{2gc_p}\left[1 - \frac{4m^2}{k} + 2\beta\frac{k-2}{k+2} + \beta^2\frac{(k-2)^2}{k(k+2)}\right] \tag{2.5}$$

Consequently, the difference between proper temperature T_e and T_0 is a function of the temperature of adiabatic compression, the exhaust velocity and of the inhomogeneity of the oncoming stream, but it does not depend upon the Prandtl number.

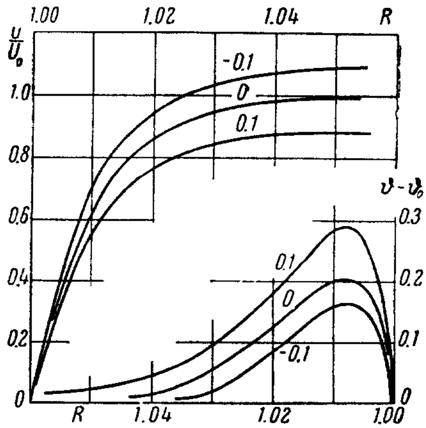
Note that for large Reynolds numbers $|k| \gg 1$, therefore expression (2.5) may be approximately written in the form

$$T_e - T_0 \approx \frac{U_0^2}{2gc_p}(1+\beta)^2 \tag{2.6}$$

i.e. a relatively small inhomogeneity of the oncoming stream may have a substantial influence on the proper temperature of the wall.

The cooling problem. The boundary condition at the wall in this case is given by

$$T = T_1 \quad \text{or} \quad \vartheta = \vartheta_1 \text{ where } R = 1 \tag{2.7}$$



and the solution of Equation (2.1), which satisfies this condition, has the form

$$\vartheta - \vartheta_0 = (\vartheta_1 - \vartheta_0) R^{\sigma k} - \frac{2\sigma\beta^2}{4 - k\sigma}(R^4 - R^{\sigma k}) - \frac{4\sigma m^2}{2 + k\sigma}(R^{-2} - R^{\sigma k}) +$$
$$+ \frac{8k\sigma\beta(1+\beta)}{(k+2)(k+2-k\sigma)}(R^{k+2} - R^{\sigma k}) - \frac{\sigma(1+\beta)^2}{2-\sigma}(R^{2k} - R^{\sigma k}) \qquad (2.8)$$

In conclusion, we shall investigate the question of the direction of thermal conductivity, i.e. whether the heat is transferred from the wall to the moving fluid or vice versa. For this we shall determine the sign of the temperature gradient near the wall. After differentiating Equation (2.8) with respect to R and substituting $R = 1$ into the equation thus obtained, we have

$$\left.\frac{d\vartheta}{dR}\right|_{R=1} = \sigma k(\vartheta_1 - \vartheta_0) - 2\sigma\beta^2 + 4\sigma m^2 + \frac{8\sigma k}{k+2}\beta(1+\beta) - \sigma k(1+\beta)^2 \qquad (2.9)$$

If $d\vartheta/dR < 0$ for $R = 1$, then the heat transfer is from the cylinder to the fluid, and, conversely, if $d\vartheta/dR > 0$ for $R = 1$, then the heat transfer is from the fluid to the cylinder. Consequently, the following inequality serves as a criterion of the heat flow from the heated cylinder to the moving fluid (or vice versa):

$$T_1 - T_0 \gtrless \left[\frac{2\beta^2}{k} - \frac{4m^2}{k} - \frac{8\beta(1+\beta)}{k+2} + (1+\beta)^2\right]\frac{U_0^2}{2gc_p} \qquad (2.10)$$

or, using (2.5), as in the case of a disk immersed in a potential stream, we have

$$T_1 - T_0 \gtrless T_e - T_0 \quad \text{(heated wall} \rightleftarrows \text{fluid)} \qquad (2.11)$$

From the expressions (2.11) and (2.6) it follows that the inhomogeneity of an oncoming stream may substantially increase or, conversely, decrease the cooling action of the fluid flowing past a cylinder. In essence, everything will depend on the sign of the inhomogeneity of the oncoming stream β. For $\beta < 0$ the temperature gradient and the thickness of the thermal boundary layer will be smaller than in the case of the cylinder immersed in a potential stream, and, conversely, for $\beta > 0$ they will be greater. This may be illustrated by the distribution of the temperatures $\vartheta - \vartheta_0$ in the boundary layer, shown in the lower part of the figure, calculated from Formula (2.8) for $\varkappa = 10^{-4}$, $R = 10^{-6}$, $\vartheta_1 = \vartheta_0$ and $\sigma = 0.72$ for the three values of vorticity of the oncoming stream $\beta = -0.1$, 0 and 0.1.

Translated by J.R.W.